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A symmetry extension of Maxwell's rule for rigidity of frames

P.W. Fowler[a], S.D. Guest[b,*]

[a]Department of Chemistry, University of Exeter, Stocker Road, Exeter EX4 4QD, UK
[b]Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK

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Abstract

A symmetry extension of Maxwell's rule for rigidity of frames is found. This rule subsumes and strengthens Maxwell's 1864 rule by requiring that the internal and external degrees of freedom of a pin-jointed structure are not only numerically equal, but are also equisymmetric. A number of special cases of Maxwell's original rule are studied to show the improved insight that the symmetry-adapted version can give. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction: the algebraic rule

In 1864 James Clerk Maxwell published an algebraic rule setting out a condition for a pin-jointed frame composed of b rigid bars and j frictionless joints to be both statically and kinematically determinate i.e. 'just stiff' (Maxwell, 1864). The number of bars needed to stiffen a three-dimensional frame free to translate and rotate in space as a rigid body, is

$$b = 3j - 6 \tag{1a}$$

The physical reasoning behind the rule is clear: each added bar links two joints and removes at most one internal degree of freedom. It is trivial to modify Maxwell's rule for other simple cases: for a three-dimensional frame fixed to supports,

$$b = 3j; \tag{1b}$$

for a two-dimensional frame free to translate and rotate in plane,

* Corresponding author. Fax: 01223 332 662.
E-mail address: sdg@eng.cam.ac.uk (S.D. Guest)

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$$b = 2j - 3; \tag{1c}$$

for a frame fixed to supports and confined to a plane,

$$b = 2j. \tag{1d}$$

All four of these cases are covered by a formulation such as (1) with appropriate values of t and Δ:

$$b = tj - \Delta \tag{1}$$

As Maxwell himself noted, (1) is a necessary but not in general a sufficient condition for establishing determinacy. A full account of the degrees of freedom of the frame must allow for the possibility of *states of self stress* (bar tensions in the absence of external load) and *mechanisms* (displacements of joints without any bar extensions). The full inventory of degrees of freedom of the frame can therefore be written as an extended Maxwell's rule (Calladine, 1978)

$$b - tj + \Delta = s - m \tag{2}$$

where s and m count the states of self stress and mechanisms, respectively, and can be determined by finding the rank of the equilibrium matrix that describes the frame in a full structural analysis (Pellegrino and Calladine, 1986). The nature of Maxwell's rule as necessary rather than sufficient is made clear by (2): vanishing of the LHS implies only that the numbers of mechanisms and states of self stress are equal, not that they are both zero.

It is the purpose of the present note to point out that a more specific form of both Maxwell's rule, and the extended Maxwell's rule, can be found by considering (2) and hence (1) in the light of not only the *numbers* of structural components, forces and displacements of a frame, but also their *symmetries*. Given the *reducible representations* of the bar extensions and joint displacements of a symmetric frame, which are easily calculated by standard methods (Wilson, Decius and Cross, 1955), the algebraic formula (2) appears as an aspect of a more general relation which can place useful limitations on possible indeterminacies. This extension of an algebraic to a group-theoretical relation parallels recent developments in a chemical context, where Euler's famous polyhedral theorem relating numbers of vertices, edges and faces has been shown to have powerful symmetry counterparts which clarify the description of molecular vibration and chemical bonding in cages (Ceulemans and Fowler, 1991; Fowler and Ceulemans, 1995).

The symmetry extension of Maxwell's rule does not attempt to provide a complete structural analysis—if this is required, then the methods described by Pellegrino and Calladine (1986), or, in a symmetry context, Kangwai and Guest (1999b), should be explored. These methods are computer-based; the symmetry extension of Maxwell's rule is by contrast suitable for a preliminary back-of-the-envelope calculation, more specific than, but in the same spirit as Maxwell's original rule. The new rule is useful, e.g. to elucidate the symmetry-based properties of the structures described in Section 3, or to provide an initial explanation of the paradoxical properties of the symmetric structures described in Kangwai and Guest (1999a).

2. A symmetry version of the rule

Eqns (1) and (2) relate the total number of external and internal degrees of freedom. The aim of this section is to find a simple symmetry-adapted version of these equations, which relates the numbers of external and internal degrees of freedom that have a particular type of symmetry. This paper outlines

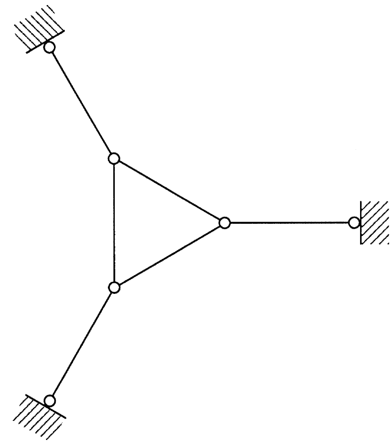

Fig. 1. Pin-jointed structure in 2-dimensional space with $\mathbf{C}_{3v}$ symmetry.

three ways of finding these numbers; they are progressively simpler, and the final version is an easy to use extension of Maxwell's original rule written in the language of representations.

2.1. A brute-force approach

One possible approach is the brute-force method described by Kangwai and Guest (1999b) where a complete symmetry-adapted coordinate system is found for the entire structure. Consider, for example, the 2D structure shown in Fig. 1. Considered in 3D, the structure has $\mathbf{D}_{3h}$, but if we consider only motions restricted to the plane, this full point group can be replaced by its essential subgroup $\mathbf{C}_{3v}$ to shorten calculations. This structure satisfies Maxwell's rule, (1d), and hence the number of states of self-stress and mechanisms must be equal.

Full symmetry-adapted coordinate systems were found for the example structure by Kangwai and Guest (1999b) for both the external degrees of freedom used for representing e.g., nodal forces or displacements, and the internal degrees of freedom used for representing e.g., bar tensions or extensions. This analysis showed that the structure has one external and two internal degrees of freedom with the full symmetry of the structure, corresponding to irreducible representation A_1, and hence one state of self-stress. It also showed that the structure has one external, and no internal degrees of freedom, with the rotational, but not reflection, symmetry of the structure, corresponding to irreducible representation A_2, and hence one mechanism.

While this brute-force approach undoubtedly works, it gives the number of external and internal degrees of freedom with particular symmetry as a final result of the analysis, and not as a simple preliminary calculation in the spirit of Maxwell's original rule.

2.2. An approach based on the traces of representation matrices

Consider the Cartesian coordinate system attached to the example structure in Fig. 2. Each of the symmetry operations in the symmetry group $\mathbf{C}_{3v}$ can now be written as a *reducible* matrix representation. For example, rotation by $2\pi/3$ would be written as:

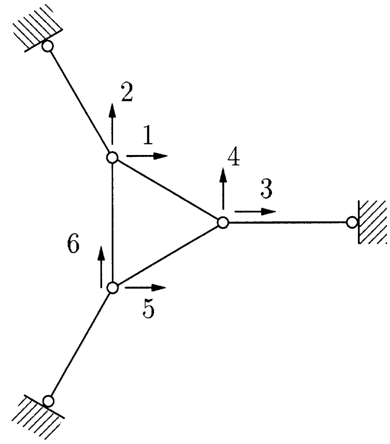

Fig. 2. A cartesian coordinate system for the external space.

$$\begin{bmatrix} 0 & 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 0 & \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Group Representation Theory shows that, had we used the correct coordinate system, all of these representations would have had a block-diagonal form, where the blocks were made up of the *irreducible representations* given in Table 1. The number of degrees of freedom of the external coordinate system with each type of symmetry can be found by counting these blocks.

The key to making further progress is to realise that a change in coordinate system cannot change the trace, or *character* of a representation matrix. The character of each of the reducible matrix representations can be calculated from matrices such as the one above. The character for each of the irreducible matrix representations are given in a *character table* such as Table 2. As can be seen, some symmetry operations always share the same character, and this properly divides the operations into symmetry *classes*. It now becomes clear how many copies of each irreducible representation must be

Table 1
A set of irreducible representations of $\mathbf{C}_{3v}$. The 1-dimensional representations A_1 and A_2 are unique. The two-dimensional representation E has a degree of arbitrariness, in that it depends on the choice of an orientation for the two in-plane x and y unit vectors

$\mathbf{C}_{3v}$	E	C_3	C_3^2	σ_a	σ_b	σ_c
A_1	1	1	1	1	1	1
A_2	1	1	1	−1	−1	−1
E	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

Table 2
Character table for $\mathbf{C}_{3v}$

$\mathbf{C}_{3v}$	E	$2C_3$	$3\sigma_v$
A_1	1	1	1
A_2	1	1	−1
E	2	−1	0

present if the characters of the representations under all symmetry operations are to match. Consider, for example, the external coordinate system shown in Fig. 2. The character of the representation under the identity is 6, under rotations of $2\pi/3$, $4\pi/3$ it is 0, and under any reflection it is 0. These can then be written as an array of three numbers, $\Gamma(e) = \{6,0,0\}$. Considering Table 2, it is clear that this can only correspond to the irreducible representations $\Gamma(e) = A_1 + A_2 + 2E = \{1,1,1\} + \{1,1,-1\} + 2 \times \{2,-1,0\} = \{6,0,0\}$, which agrees with the dimensions found by the brute force method in Kangwai and Guest (1999b).

A more systematic way of deriving the reduction to irreducible representations is by projection. If the character of a reducible representation of a symmetry operation R is written $\chi^{\mathrm{red}}(R)$, and the corresponding character of a given irreducible representation μ is written as $\chi^{\mu}(R)$, then it can be shown that the number of times, a_μ, that the irreducible representation Γ_μ appears is

$$a_\mu = \frac{1}{g}\sum_R \chi^{\mathrm{red}}(R)\chi^{\mu*}(R)$$

where * represents the complex conjugate, and g is the order (number of operations) of the group.

A similar method can be implemented for the internal coordinate system. A natural internal coordinate system is shown in Fig. 3. Again, each of the symmetry operations can be written as a matrix representation, but in this case the representations are simply permutation matrices. This makes the characters of the representation very easy to find—it is simply the number of bars remaining unshifted under a particular symmetry operation, as only when a bar is left in place can a 1, rather than a 0, appear on the diagonal.

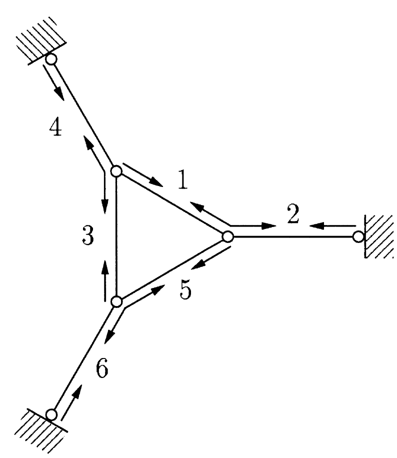

Fig. 3. A natural coordinate system for the internal space.

The characters of the representations for the internal coordinate system are therefore $\Gamma(b) = \{6,0,2\}$. Considering Table 2, it is clear that this must be made up of the irreducible representations $\Gamma(b) = 2A_1 + 2E$, which again agrees with the results of the brute-force method in Kangwai and Guest (1999b).

We are now in a position to write a first attempt at a symmetry extension of Maxwell's law. For a frame to be just stiff, the following is a necessary condition

$$\Gamma(e) = \Gamma(b).$$

Not only must the number of external and internal degrees of freedom be equal, but so must their symmetries. Clearly this is not satisfied for our example structure.

While this matrix method works, there are two improvements that can be made. First, the methods seems to require the finding of a representation matrix for each class of symmetry operation in order to find the character of the external representations. It will be seen that this can be easily improved so that it requires little more than the counting required for the internal case. Secondly, the current treatment has only considered structures that are firmly anchored to their foundations. Maxwell's original rule was for a body free from foundations, but to derive the symmetry equivalent requires a way of finding the symmetry properties of free-body motion. Both of these problems will be solved in the next section.

2.3. *A simplified way of finding the character of representations*

The representations of symmetry operations for an external coordinate system can be considered in two parts. Firstly, there is a block permutation which switches the nodes between different locations; secondly, each of the blocks is a local (2×2) or (3×3) transformation matrix representing the effect of a symmetry operation on the coordinate system for a single central point.

A contribution to the character of a representation can, therefore, be non-zero only if a node remains unmoved by a symmetry operation. The character of the representation will then be the character of the symmetry operation acting on the coordinate system, multiplied by the number of nodes unmoved by the symmetry operation.

For the example structure, the permutation character of the joints is $\Gamma(j) = \{3,0,1\}$: all nodes remain unmoved by the identity, no nodes remain unmoved by the rotations, and one node remains unmoved by any given reflection. The character for the symmetry transformations of the coordinates of a single point in 2D are $\Gamma(T) = \{2,-1,0\}$. The external degrees of freedom can therefore be written

$$\Gamma(e) = \Gamma(j) \otimes \Gamma(T)$$

where $\otimes$ means a class-by-class multiplication of corresponding characters. For the example structure

$$\Gamma(e) = \{3,0,1\} \otimes \{2,-1,0\} = \{6,0,0\}$$

as before.

The way to handle rigid body motions is now also clear. A representation of a rigid body translation is simply the $\Gamma(T)$ given above. In a similar way, it is possible to find the characters $\Gamma(R)$ for the rigid body rotations, of which there is 1 in 2D, and 3 in 3D. Both $\Gamma(T)$ and $\Gamma(R)$ can be read off from tables of the point groups (Atkins, Child and Phillips, 1970).

As a simple example, consider the triangular structure shown in Fig. 4—the original example structure minus the bars connecting it to the foundation. The representation of the possible motions of its nodes is identical to the original example structure, $\Gamma(e) = A_1 + A_2 + 2E_2$, but this now includes the rigid body motions. As calculated above, $\Gamma(T) = \{2,-1,0\} = E$, and in 2D the rigid-body rotation has $\Gamma(R) = \{1,1,-1\} = A_2$, as a rigid body rotation is preserved by the identity and rotation, but reversed by reflection. Thus the external degrees of freedom, after removal of the rigid body motions are

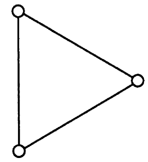

Fig. 4. Free-floating pin-jointed structure in 2-dimensional space with $\mathbf{C}_{3v}$ symmetry.

$\Gamma(j)\otimes\Gamma(T)-\Gamma(T)-\Gamma(R)=A_1+E$. The characters of the internal coordinate system of the new structure are $\Gamma(i)=\{3,0,1\}=A_1+E$. Thus, this structure satisfies a necessary condition for being just stiff—the external and internal degrees of freedom have the same symmetry.

We are now in a position to re-write Maxwell's eqns (1) in a symmetry adapted form. The internal degrees of freedom must be equisymmetric with the external degrees of freedom, after the removal of any rigid body motions. In the language of representations the direct equivalent of Maxwell's original rule, (1a), becomes

$$\Gamma(b)=\Gamma(j)\otimes\Gamma(T)-\Gamma(T)-\Gamma(R). \tag{3a}$$

Equally, it is possible to write three further equations corresponding to (1b)–(1d)

$$\Gamma(b)=\Gamma(j)\otimes\Gamma(T) \tag{3b}$$

$$\Gamma(b)=\Gamma(j)\otimes\Gamma\left(T_{\parallel}\right)-\Gamma\left(T_{\parallel}\right)-\Gamma(R_{\perp}) \tag{3c}$$

$$\Gamma(b)=\Gamma(j)\otimes\Gamma\left(T_{\parallel}\right) \tag{3d}$$

$\Gamma(T_{\parallel})$ and $\Gamma(R_{\perp})$ are the components of $\Gamma(T)$ and $\Gamma(R)$ that represent the two translations and one rotation remaining to a body confined to the plane. By analogy with (1), a summary form of (3a)–(3d) is

$$\Gamma(b)=\left\{\Gamma(j)\otimes\Gamma(t)-\Gamma(\Delta)\right\} \tag{3}$$

with appropriate definitions of $\Gamma(t)$ and $\Gamma(\Delta)$.

As with Maxwell's original rule, (3) is a necessary, but not sufficient condition for establishing determinacy. Again, a full account of the degrees of freedom must allow for the possibility of states of self stress, and mechanisms. The symmetry version of the extended Maxwell's rule therefore becomes

$$\Gamma(b)-\left\{\Gamma(j)\otimes\Gamma(t)-\Gamma(\Delta)\right\}=\Gamma(s)-\Gamma(m) \tag{4}$$

where $\Gamma(m)$ is the representation of any mechanisms, and $\Gamma(s)$ is the representation of any states of self stress.

Eqn (1) is revealed as the character $\chi(E)$ of (3) under the identity operation, and (3) would collapse to (1) in the absence of structural symmetry. Eqn (3) is again only a necessary condition: if it is satisfied, then *either* the frame is rigid ($s=m=0$) *or* those mechanisms and states of self stress that do exist are equisymmetric ($\Gamma(s)=\Gamma(m)$).

Eqn (3) is more restrictive than (1) in that the matching of LHS and RHS must apply block-by-block to each irreducible representation Γ_a of the point group G. Thus, if the decompositions

$$\Gamma(b)=\sum_a b_a\Gamma_a$$

$$\Gamma(j) \otimes \Gamma(t) - \Gamma(\Delta) = \sum_a j'_a \Gamma_a$$

have been calculated, a necessary condition for rigidity is $b_a = j'_a$ for all a. b_a, j'_a are the dimensions of the blocks of a symmetry-adapted equilibrium or compatibility matrix (Kangwai and Guest, 1999b). In any block where $b_a > j'_a$ there are at least $b_a - j'_a$ states of self stress of symmetry Γ_a and conversely in any block where $b_a < j'_a$ there are at least $j'_a - b_a$ mechanisms of symmetry Γ_a. A structure may thus obey Maxwell's algebraic rule, (1), but be revealed as indeterminate by examination of the blocks Γ_a, as was seen for the structure in Fig. 1. It can be seen that (3) is a necessary, but not sufficient, condition because of the possibility of these blocks being rank-deficient. Rank-deficiency of the blocks signals the presence of equisymmetric states of self-stress and mechanisms, as allowed in (4).

3. Examples

This section will consider a number of the examples mentioned by Calladine (1978) as special cases of Maxwell's rule, to examine whether the new symmetry version gives improved insight into these structures.

3.1. Unsupported planar frames

A classic problem case for Maxwell's algebraic rule is the pair of frames (a) and (b) illustrated in Fig. 5. Both have $j = 6$ and $b = 2j - 3 = 9$ and so nominally obey Maxwell's algebraic rule, but (a) is rigid in the plane whereas (b) is part mechanism and part redundant.

Considering only motions in the plane, frame (a) has $\mathbf{C}_2$ symmetry with operations $\{E, C_2\}$; the character table for $\mathbf{C}_2$ is shown in Table 3. The RHS of (4c) is

	E	C_2	
$\Gamma(j)$	6	0	
$\otimes\Gamma(T_\parallel)$	2	-2	
$=$	12	0	
$-\Gamma(T_\parallel)$	-2	2	
$-\Gamma(R_\perp)$	-1	-1	
$=$	9	1	$=5A+4B$

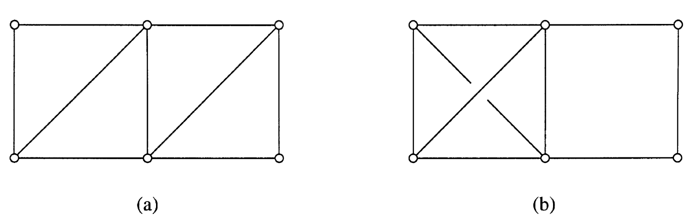

Fig. 5. Two plane frames which satisfy Maxwell's original rule. (a) is simply stiff, (b) is part redundant and part mechanism.

Table 3
Character table for $\mathbf{C}_2$

$\mathbf{C}_2$	E	C_2
A	1	1
B	1	−1

Similarly for the LHS of (4c)

$$\begin{array}{r|cc} & E & C_2 \\ \hline \Gamma(b) & 9 & 1 \end{array} = 5A + 4B$$

Thus (a) obeys Maxwell's rule in both algebraic and symmetric forms.

Considering only motions in the plane, frame (b) has $\mathbf{C}_s$ symmetry with operations $\{E, \sigma\}$; the character table for $\mathbf{C}_s$ is shown in Table 4. The RHS of (4c) is

$$\begin{array}{r|cc} & E & \sigma \\ \hline \Gamma(j) & 6 & 0 \\ \otimes\Gamma(T_{\parallel}) & 2 & 0 \\ \hline = & 12 & 0 \\ & & \\ -\Gamma(T_{\parallel}) & -2 & 0 \\ -\Gamma(R_{\perp}) & -1 & +1 \\ \hline = & 9 & 1 \end{array} = 5A' + 4A''$$

Similarly for the LHS of (4c)

$$\begin{array}{r|cc} & E & \sigma \\ \hline \Gamma(b) & 9 & 3 \end{array} = 6A' + 3A''$$

The conclusion is that (b) must have at least one state of self stress (of symmetry A') and one mechanism (of symmetry A''). These are in fact the only indeterminacies of the system and so the description arising from (4) characterises fully both frames (a) and (b).

These examples also illustrate the point alluded to earlier. It is not always necessary to use the full

Table 4
Character table for $\mathbf{C}_s$

$\mathbf{C}_s$	E	σ
A'	1	1
A''	1	−1

point group of a system to extract this information. The full symmetries of frames (a) and (b) are $\mathbf{C}_{2h}$ and $\mathbf{C}_{2v}$ respectively, but in-plane motions are totally symmetric under the horizontal mirror of $\mathbf{C}_{2h}$ and $\mathbf{C}_{2v}$, and so all useful information is given by calculations within the appropriate subgroups. The same motivation underlies our use of $\mathbf{C}_{3v}$ rather than $\mathbf{D}_{3h}$ symmetry in Sections 2.1 to 2.3.

3.2. Deltahedral frames

One well known special case of Maxwell's algebraic rule is the deltahedron, a polyhedron made up solely of triangular faces. Given Euler's theorem connecting the number of edges e, vertices v and faces f of a polyhedra, $v+f=e+2$, and the identity $3f=2e$ that applies when all faces are triangular, the vertex (joint) and edge (bar) counts of a deltahedron must satisfy $e=3v-6$ and hence (1a). This is a necessary condition for rigidity of the deltahedral frame. Convexity is one example of an additional condition that would be sufficient to enforce rigidity (Cauchy, 1813), though the existence of rigid 'dimpled' deltahedra shows that it is not the only one. The symmetry counterpart of (1a) in terms of permutation representations of vertices and edges is

$$\Gamma(e)=\Gamma(v)\otimes\Gamma(T)-\Gamma(T)-\Gamma(R)$$

which is a theorem that has been proved in the context of vibrations of deltahedral molecules (Ceulemans and Fowler, 1994). Eqn (3) is also therefore satisfied by all deltahedral frames.

For an example of how the symmetry theorem applies to non-rigid deltahedra, consider some modifications of the icosahedron considered by Calladine (1979). The bars of a regular convex icosahedral frame span

$$\Gamma(b)=A_g+G_g+2H_g+T_{1u}+T_{2u}+G_u+H_u.$$

The internal degrees of freedom $\Gamma(j)\times\Gamma(T)-\Gamma(T)-\Gamma(R)$ span the self-same representation and indeed the frame is rigid.

If now one vertex of the icosahedron is pushed radially inwards, with simultaneous lengthening of the five bars in contact with it, the symmetry drops to $\mathbf{C}_{5v}$ and the bar representation becomes

$$\Gamma(b)=5A_1+A_2+6E_1+6E_2$$

which is still equisymmetric with $\Gamma(j)\otimes\Gamma(T)-\Gamma(T)-\Gamma(R)$ and so obeys (3). However, at the point on the distortion pathway where the depressed joint becomes coplanar with its five neighbours, the structure gains a mechanism: the central joint in the large planar face may move infinitesimally in and out without requiring any change to first order in bar lengths. The mechanism is totally symmetric under the operations of $\mathbf{C}_{5v}$, and by (4) must be accompanied by a state of self stress of the same symmetry, $\Gamma(m)=\Gamma(s)=A_1$.

If the central vertex is pushed in further, to give a pentagonally pyramidal dimple, the bars can shorten again, the symmetry remains $\mathbf{C}_{5v}$, and (4) remains true, though s and m have both dropped back to zero. Thus neither algebraic nor symmetry versions give a hint of the indeterminacy that arises for special geometries within the same overall icosahedral topology. To find this a full structural analysis is needed, to identify the rank-deficient blocks of the equilibrium matrix.

3.3. Tensegrity structures

Calladine (1978) used the extended Maxwell's rule (2) to analyse a number of well-known tensegrity

Table 5
Symmetry analysis of well-known tensegrities

Marks' (1960) designation	Description	$b-3j+6=$ $s-m$	Symmetry	$\Gamma(b)$	$\Gamma(j)\otimes\Gamma(T)-\Gamma(T)-\Gamma(R)$	$\Gamma(s)-\Gamma(m)$	Note
K8	Tensegrity Tetrahedron	-6	**T**	$3A+3E+5T$	$3A+3E+7T$	$-2T$	1(a)
K11	Tensegrity Tricontahedron	-54	**I**	$3A+7T_1+9T_2+12G+15H$	$3A+5T_1+5T_2+8G+11H$	$-2T_1-4T_2-4G-4H$	
K9	Tensegrity Icosahedron	0	$\mathbf{T}_h$	$2A_g+2E_g+3T_g+A_u+E_u+4T_u$	$2A_g+2E_g+3T_g+A_a+E_u+4T_u$	0	1(b)
K10	Tensegrity Vector Equilibrium	0	$\mathbf{S}_6$	$5A_g+5E_g+5A_u+5E_u$	$5A_g+5E_g+5A_u+5E_u$	0	1(c)
K8A	Tensegrity Octahedron	3	$\mathbf{O}_h$	$2A_{1g}+2E_g+T_{2g}+T_{1u}+T_{2u}$	$A_{1g}+E_g+T_{2g}+T_{1u}+T_{2u}$	$A_{1g}+E_g$	2

Notes.
1. Calladine established by experimentation that there was a single state of self-stress for these three cases, and hence:
(a) $\Gamma(s)=A$, $\Gamma(m)=A+2T$.
(b) $\Gamma(s)=\Gamma(m)=$ either A_g or A_u.
(c) $\Gamma(s)=\Gamma(m)=$ either A_g or A_u.
2. The outer wires of the octahedron form a deltahedron, necessarily determinate, and hence $\Gamma(s)-\Gamma(m)$ is exactly the representation of the additional diagonal bars. Calladine pointed out that there are no mechanisms, and so $\Gamma(s)=A_{1g}+E_g$.

structures that are illustrated in the book by Marks (1960). Table 5 reproduces the salient features of Calladine's results, and adds the results of an analysis based on the symmetry Maxwell rule, (4).

4. Conclusion

A symmetry extension of Maxwell's rule can give more specific information on rigidity, states of self stress and mechanisms than the traditional algebraic formulation. It uses the tools of applied point group theory, which were not available when Maxwell enunciated his rule, but otherwise follows his line of reasoning exactly. The new, stronger version of the rule is easy to apply, requiring only consultation of a character table and counting of structural components shifted and unshifted by symmetry operations.

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